Minoration of the complexity function associated to a translation on the torus

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Abstract

We show that the word complexity function $p_k(n)$ of a piecewise translation map conjugated to a minimal translation on the torus $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ is at least $kn + 1$ for every integer n.

1 Introduction

Symbolic dynamics is a part of dynamics which studies the interaction between dynamical systems and combinatorial properties of sequences defined over a finite number of symbols. The collection of these symbols is called *the alphabet*. Here we are interested in a minimal translation on the torus $\mathbb{T}^k$, *i.e* almost every point has a dense orbit under the action of the translation. We consider a finite partition of the torus, and the coding of a translation related to this partition. An orbit is coded by an one-sided infinite sequence over a finite alphabet, the orbit is called *an infinite word*. The complexity of an infinite word is a function defined over integers $n \in \mathbb{N}$: it is the number of different words of n digits which appear in such a sequence. For a minimal dynamical system, when the partition consists of "reasonable" sets, the complexity does not depend on the orbit. Here we search partitions which minimize the complexity function.

The case of the circle (identified to $[0, 1]$) is known since the work of Hedlund and Morse: a translation by angle α is coded by the intervals $[0, 1 - \alpha)$ and $[1 - \alpha, 0)$. Then the complexity function is equal to $n + 1$ if $\alpha \notin \mathbb{Q}$, see [MH40]. Moreover a sequence of complexity $n + 1$ can be characterized as a coding of a translation by an irrational angle on the circle, see [CH73] and[PF02].

For the two dimensional torus, the first result was made for a particular translation: there exists a translation and a partition which have complexity $2n + 1$. This is a famous result of Rauzy, the partition of the torus is called the Rauzy fractal and each element has a fractal boundary, see [Rau82]. Some bound on the complexity function are given for any translation if the partition is made of polygons. The first one has been made in [AMST94] for a partition in three rhombi. This result has been generalized to any dimension in [Bar95]. In [Bed03] a correction of the proof in the two dimensional cases was made and in [Bed09] an improvement of the result in any dimension was done for the same partition. Finally in [Che09] the growth of the complexity was computed for every polygonal partition in $\mathbb{T}^2$. In this case the complexity is quadratic in n. In [Ber12], the second author shows that for a translation on a two dimensional torus, the complexity function related to a partition cannot be less than $2n + 1$. This bound is sharp, by Rauzy's result.

Here we prove a similar result in the k dimensional torus.

Theorem 1.1. *Let $k \geq 1$ and $m \geq 1$ be two integers, let $\mathbf{a}$ be a vector in $\mathbb{R}^k$ such that the translation by $\mathbf{a}$ on the torus $\mathbb{T}^k$ is minimal. Let $(T, \mathcal{D}_1, \dots, \mathcal{D}_m)$ be a piecewise translation associated to this translation. Then the complexity function of the piecewise translation fulfills*

$$\forall n \geq 1, \quad p_k(n) \geq kn + 1.$$

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The complexity function is related to a topological invariant of dynamical systems: the topological entropy. This notion is used to classify systems of zero topological entropy. There are few systems for which the complexity can be computed. Recently Host, Kra and Maass give a lower bound in [HKA13], of the complexity of a class of dynamical systems called nilsystems. These systems are closed to rotations.

2 Examples

We begin by a list of examples which show that the bound on the complexity in Theorem 3.3 is sharp. We refer to [PF02] for a complete background on combinatorics on words.

2.1 Background on Substitutions

To present the examples we need to give some background on Combinatorics on Words. Consider an alphabet over k letters $\{a_1, a_2, \dots, a_k\}$ and the free monoid $\{a_1, a_2, \dots, a_k\}^*$. A morphism of the free monoid is called a *substitution*. For example consider

$$\sigma_k = \sigma : \begin{cases} a_1 \mapsto a_1 a_2 \\ a_2 \mapsto a_1 a_3 \\ \vdots \\ a_k \mapsto a_1. \end{cases}$$

This substitution is called *k-bonacci* substitution. The sequence $(\sigma^n(a_1))_{n\in\mathbb{N}}$ converges for the product topology to a fixed point w_k of σ:

$$w_k = a_1a_2a_1a_3a_1a_2a_1a_4 \dots a_1a_ka_1 \dots, \sigma(w_k) = w_k$$

The abelianization of the monoid is $\mathbb{Z}^k$. We can represent it in $\mathbb{R}^k$ with a basis where each coordinate vector represents either $a_1, \dots, a_{k-1}$ or a_k. The abelianization of σ is a linear morphism of $\mathbb{Z}^k$ with matrix

$$M_\sigma = \left(\begin{array}{c|c} 1 \ \cdots \ 1 & 1 \\ \hline & 0 \\ I_k & \vdots \\ & 0 \end{array}\right), \text{ where } I_k \text{ is the identity matrix of } \mathbb{R}^{k-1}.$$

The abelianization of u is a broken line in $\mathbb{Z}^k$ denoted δ.

This matrix has one real eigenvalue of modulus bigger than 1. The eigenspace is of dimension one, and is called H_e. The other eigenvalues are complex number if $k \geq 3$, we denote by H_c a real hyperplane orthogonal to H_e. Consider the projection on H_c parallel to H_e. Denote $\mathcal{R}_{k-1}$ the closure of the projection of vertices of δ. This set is the *Rauzy fractal* associated to the substitution. There is a natural partition of this set in k subsets defined as follows: First the vertices of δ can be split in k classes. One made of vertices followed by an edge in the a_1 direction, one made of vertices followed by an edge in the a_2 direction, and so on. The closure of the projection of each family of vertices gives us a subset of R_{k-1} denoted $R_{k-1}(a_i), i = 1 \dots k$.

$$\mathcal{R}_{k-1} = \bigcup_{i=1}^{k} \mathcal{R}_{k-1}(a_i).$$

Theorem 2.1. *[Mes98] For every integer $k \geq 1$ the Rauzy fractal $\mathcal{R}_k$ is a fundamental domain of the torus $\mathbb{T}^k$. Let u be the projection of one basis vector of $\mathbb{R}^{k+1}$ on H_c. Then the translation by vector u on the torus $\mathbb{T}^k$ is minimal, and the complexity of this map related to the partition of $\mathcal{R}_k$ has complexity*

$$p_k(n) = kn + 1.$$

2.2 Examples of substitutions

We list some examples of k-bonacci substitutions.

- For $k = 2$, $R_1 = [0, 1)$, the translation is $x \mapsto x + a \quad [1]$ for $a = \varphi$ the golden mean. The partition is $D_1 = [0, 1 - a), D_2 = [1 - a, 1)$. The fixed point w_2 is called *Fibonacci word*.
- For $k = 3$, w_k is called *Tribonacci word*. The partition is made of fractal sets.

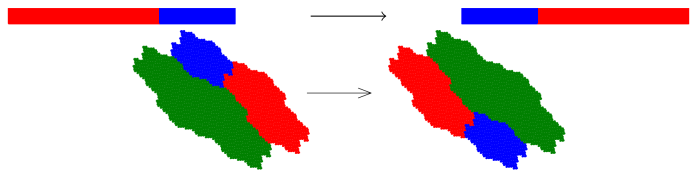


Figure 1: Representation of the domain exchange associated to the k-bonacci word for $k = 1$ and 2.

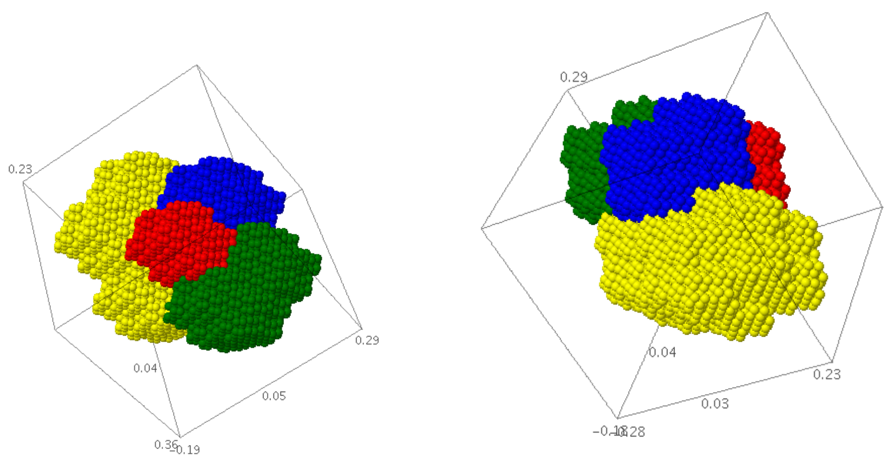


Figure 2: Representation of the Rauzy fractal associated to the 4-bonacci substitution.

2.3 Example of a translation

Here we present an example of translation on the torus $\mathbb{T}^2$. The fundamental domain of the torus is an hexagon with identified opposite edges, and the partition is made of three rhombi. The fundamental domain depends on the vector $\mathbf{a}$ of the translation:

$$\begin{array}{ccc} \mathbb{T}^2 & \to & \mathbb{T}^2 \\ m & \mapsto & m + \mathbf{a} \end{array}$$

In this case the complexity function is quadratic in n:

Theorem 2.2 ([Bed03]). *For almost all $\mathbf{a} \in \mathbb{T}^2$ the complexity function of the translation related to this partition is:*

$$p_2(n) = n^2 + n + 1.$$

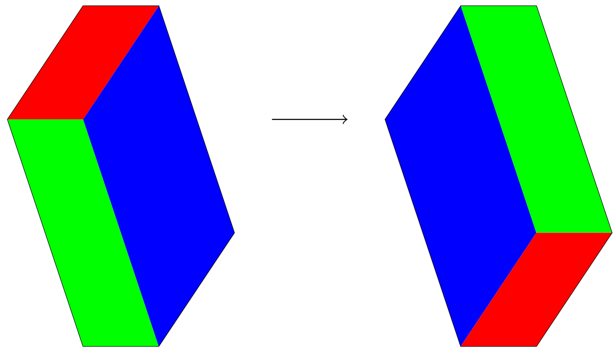


Figure 3: Translation on the two dimensional torus.

3 Notations and statement of the result

Let $k \geq 2$ be an integer, let $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{R}^k$ be a vector and λ be the Lebesgue measure on $\mathbb{R}^k$. The *translation on the torus* $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ by $\mathbf{a}$ is the map

$$\begin{array}{ll} \mathbb{T}^k & \to \mathbb{T}^k \\ \mathbf{x} & \mapsto \mathbf{x} + \mathbf{a}. \end{array}$$

This map is *minimal* (every orbit is dense) if and only if

$$\forall q_1, \dots, q_k,\ \forall q \in \mathbb{Q},\ q_1 a_1 + \dots + q_k a_k = q \Longrightarrow q_1 = \dots = q_k = 0.$$

For a minimal map, Lebesgue measure is the unique probability invariant measure and then, it is an ergodic measure.

A *fundamental domain* of the torus $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ is a subset D of $\mathbb{R}^k$ which periodically tiles the space $\mathbb{R}^k$ by the action of $\mathbb{Z}^k$, except maybe on a subset of $\mathbb{R}^k$ with measure zero.

Definition 3.1. A *piecewise translation map* $(T, \mathcal{D}_1, \dots, \mathcal{D}_m)$ is *associated* to the translation by the vector $\mathbf{a}$ if there exist m measurable disjoint sets $\mathcal{D}_1, \dots, \mathcal{D}_m$ of $\mathbb{R}^k$ such that

- $\mathcal{D} = \bigcup_{i=1}^{m} \mathcal{D}_i$ is a fundamental domain of the torus,
- the measurable map $T : \mathcal{D} \mapsto \mathcal{D}$ is defined by:
$$T(\mathbf{x}) = \mathbf{x} + \mathbf{a} + \mathbf{n}(\mathbf{x}),$$
- where $\mathbf{n} : \mathcal{D} \mapsto \mathbb{Z}^k$ is a measurable map such that
$$\int_{\mathcal{D}_i} \mathbf{n}(\mathbf{x})\ \mathsf{d}\lambda(\mathbf{x}) = \lambda(\mathcal{D}_i)\mathbf{r}_i \text{ where } \mathbf{r}_i \in \mathbb{Q}^k. \tag{1}$$

The *coding* of the orbit of a point $\mathbf{x} \in \mathcal{D}$ under T is defined by

$$\mathsf{Cod} : \mathcal{D} \mapsto \{1, \cdots, m\}^{\mathbb{N}}$$

$$\mathsf{Cod}(\mathbf{x})_n = i \Longleftrightarrow T^n \mathbf{x} \in \mathcal{D}_i.$$

Remark 3.2. The preceding definition is different from the definition of a partition, due to the two last points. First the domain of $\mathcal{D}$ is not assumed to be bounded. Moreover here we do not assume that Cod is an injective map, see the work of Halmos for a general reference about the coding.

A word of length n in $\mathsf{Cod}(\mathbf{x})$ is a finite sequence of the form $(\mathsf{Cod}(\mathbf{x}))_{p\leq m\leq p+n-1}$. The *complexity function* of the piecewise translation $(T,\mathcal{D}_1,\dots,\mathcal{D}_m)$ is a map from $\mathbb{N}^*$ to $\mathbb{N}$ which associates to every integer $n\geq 1$, the number of different words of lengths n in $\mathsf{Cod}(\mathbf{x})$. This number is independent of the point $\mathbf{x}$ for a minimal rotation. Our principal result is the following:

Theorem 3.3. *Let $k\geq 1$ and $m\geq 1$ be two integers, let $\mathbf{a}$ be a vector in $\mathbb{R}^k$ such that the translation by $\mathbf{a}$ on the torus $\mathbb{T}^k$ is minimal. Let $(T,\mathcal{D}_1,\dots,\mathcal{D}_m)$ be a piecewise translation associated to this translation. Then the complexity function of the piecewise translation fulfills*

$$\forall n\geq 1,\quad p_k(n)\geq kn+1.$$

The proof is based on the following propositions proved in Section 5.2 and Section 5.3.

Proposition 3.4. *Let $k\geq 1$ and $m\geq 1$ be two integers, $\mathbf{a}$ a vector in $\mathbb{R}^k$ such that the translation by $\mathbf{a}$ on the torus $\mathbb{T}^k$ is minimal. Let $(T,\mathcal{D}_1,\dots,\mathcal{D}_m)$ be a piecewise translation associated to this translation. Then we have:*

$$m\geq k+1.$$

Proposition 3.5. *Let $k\geq 1$ and $m\geq 1$ be two integers, $\mathbf{a}$ a vector of $\mathbb{R}^k$ such that the translation by $\mathbf{a}$ on the torus $\mathbb{T}^k$ is minimal. Let $(T,\mathcal{D}_1,\dots,\mathcal{D}_m)$ be a piecewise translation associated to this translation. Then the complexity function fulfills for every integer n:*

$$p_k(n+1)-p_k(n)\geq k.$$

The lower bound on $p_k(n)$ in Theorem 3.3 is optimal, see Section 2 for examples.

4 Background on Graph theory

Before the proof of the result we give some background on Graph theory. This part will be used in the proof of Proposition 3.5. The reference for this section is [Bol98].

Let G be an oriented graph with n vertices $\mathsf{V}=(\mathsf{v}_1,\dots,\mathsf{v}_n)$ and m edges $\mathsf{E}=(\mathsf{e}_1,\dots,\mathsf{e}_m)$. Let $C_1(G)$ be the vectorial space generated by maps from edges to $\mathbb{R}$.

We will be interested by the functions $f\in C_1(G)$ such that in each vertex v,

$$\sum_{\mathsf{v}'\text{such that }\mathsf{v}\mathsf{v}'\in\mathsf{E}} f(\mathsf{v}\mathsf{v}')=\sum_{\mathsf{v}'\text{such that }\mathsf{v}'\mathsf{v}\in\mathsf{E}} f(\mathsf{v}'\mathsf{v}). \tag{2}$$

These functions generate a vectoriel space denoted by $N(G)$.

An oriented cycle L is defined by k vertices $\mathsf{u}_1,\dots,\mathsf{u}_k$ in V such that $\mathsf{u}_1=\mathsf{u}_k$ and for each $1\leq i\leq k-1$, $\mathsf{u}_i\mathsf{u}_{i+1}$ or $\mathsf{u}_{i+1}\mathsf{u}_i$ is an edge of the graph. We write $\mathsf{L}=\mathsf{u}_1\cdots\mathsf{u}_k$.

Let $\mathsf{L}=\mathsf{u}_1\cdots\mathsf{u}_k$ be an oriented cycle in G, then we define a function z_L on E by

$$z_\mathsf{L}(\mathsf{e})=\begin{cases}1 \text{ if there exist } j\in\{1,\dots,k-1\} \text{ such that } \mathsf{e}=\mathsf{u}_j\mathsf{u}_{j+1}\\ -1 \text{ if there exist } j\in\{1,\dots,k-1\} \text{ such that } \mathsf{e}=\mathsf{u}_{j+1}\mathsf{u}_j\\ 0 \quad \text{otherwise}\end{cases}$$

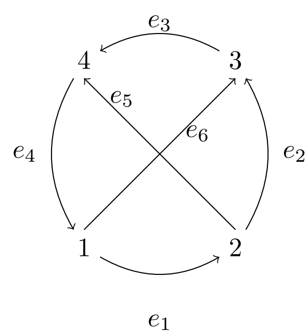
The functions z_L is called a *cycle vector* and we denote by $Z(G)$ the space generated by z_L when L runs over cycles of G. It is a subspace of $N(G)$.

Theorem 4.1. *Let G be a connected graph with n vertices and m edges. The space $Z(G)$ is equal to $N(G)$, and the dimension of the vectorial space $Z(G)$ is given by:*

$$dim(Z(G))=m-n+1.$$

The elements of a basis of $Z(G)$ are called *fundamental cycles*. We refer to [Bol98] for a proof of this result.

Example 4.2. Consider the following directed graph. The space $Z(G)$ is of dimension $6-4+1=3$. To the cycle 132 is associated the function z_{132} represented by $e_6 - e_2 - e_1$. For example, an element of $N(G)$ is the function f given by $f(e) = \begin{cases} 1 & e = e_1 \\ 0 & e = e_2 \\ 1 & e = e_3 \\ 2 & e = e_4 \\ 1 & e = e_5 \\ 1 & e = e_6 \end{cases}$. For example, by looking at vertex 1 we have: $2 = 1 + 1$.



5 Proofs of Proposition 3.4 and Proposition 3.5

5.1 A first lemma

Lemma 5.1. *Let $k, m \geq 1$ be two integers, let $\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix}$ be a vector of $\mathbb{R}^k$, $(\mathbf{n}_i)_{1 \leq i \leq m}$ m vectors of $\mathbb{Q}^k$ and let $\alpha_1, \ldots \alpha_m$ be m real numbers such that*

$$\mathbf{a} = \alpha_1 \mathbf{n}_1 + \cdots \alpha_m \mathbf{n}_m.$$

Assume $m < k$, then there exists k rational numbers $q_1, \ldots, q_k$, non all equal to zero, such that

$$a_1 q_1 + \cdots a_k q_k = 0.$$

Proof. We prove the result by induction on k. For $k = 2$, we have $m = 1$ and the hypothesis gives:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} n_1(1) \\ n_1(2) \end{pmatrix}.$$

Either $n_1(1)$ or $n_1(2)$ are non zero numbers and we obtain $a_2 n_1(1) - a_1 n_1(2) = 0$, or $n_1(1) = n_1(2) = 0$. Both cases give $\mathbf{a} = \mathbf{0}$ and the result is proved.

Assume the result is true until $k \geq 2$. The case $m = 1$ is similar to the preceding case: it suffices to consider the two first coordinates. Now assume $m > 1$, for $1 \leq i \leq k$, the i-coordinate of the vectors gives:

$$a_i = \alpha_1 n_1(i) + \cdots \alpha_m n_m(i).$$

Up to permutation, we can assume that $n_1(1)$ is non zero (if either $\mathbf{a} = \mathbf{0}$ or if there exists i such that $a_i = 0$ the result is obvious). By linear combination we have for some $2 \leq i \leq k$:

$$n_1(1) a_i - n_1(i) a_1 = \alpha_2 \begin{vmatrix} n_1(1) & n_2(1) \\ n_1(i) & n_2(i) \end{vmatrix} + \cdots + \alpha_m \begin{vmatrix} n_m(1) & n_m(1) \\ n_1(i) & n_2(i) \end{vmatrix}.$$

By induction hypothesis, there exist $(k-1)$ rational numbers not all equal to zero $q_2, \dots q_k$ such that

$$q_2\big(n_1(1)a_2 - n_1(2)a_1\big) + \cdots + q_k\big(n_1(1)a_k - n_1(k)a_1\big) = 0$$

Let us define $q_1 = -\frac{q_2 n_1(2)+\cdots+q_m n_1(m)}{n_1(1)}$. Then preceding equation can be written as

$$q_1 a_1 + q_2 a_2 + \cdots q_k a_k = 0.$$

The induction hypothesis is proven.

□

5.2 Proof of Proposition 3.4

Let $k \geq 1$ be an integer, $\mathbf{a}$ a vector of $\mathbb{R}^k$ such that the translation by $\mathbf{a}$ on the torus $\mathbb{T}^k$ is minimal. Let $(T, \mathcal{D}_1, \dots, \mathcal{D}_m)$ be a piecewise translation associated to this translation. The proof is made by contradiction. Assume $m \leq k$. For every integer $i \in \{1, \dots, m\}$, let us denote $A_i = \lambda(D_i)$ the volume of $\mathcal{D}_i$. For a function $\mathbf{f} = (f_1, \dots, f_k) \in \mathrm{L}^1_\lambda(\mathcal{D}, \mathbb{R}^k)$, Birkhoff's theorem can be applied to the ergodic map T and gives for almost every point x:

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{f}\left(T^k \mathbf{x}\right) = \left(\int_{\mathcal{D}} f_1(\mathbf{x}) d\lambda(\mathbf{x}), \dots, \int_{\mathcal{D}} f_k(\mathbf{x}) d\lambda(\mathbf{x})\right).$$

We say that x is *generic* for f if the formula is true for x. Moreover we say that x is *recurrent* if for a neighborhood V of x, there exists an integer sequence $(N_p)_{p\in\mathbb{N}}$ such that $T^{N_p}(x)$ belongs to V. Now let x be a recurrent and generic for the functions $\mathbf{n} \cdot 1_{\mathcal{D}_i}$, for $i \in \{1, \dots, m\}$ (the measure of such points is one). Let N be an integer, we obtain

$$T^N(\mathbf{x}) = \mathbf{x} + N\mathbf{a} + \sum_{k=0}^{N-1} \mathbf{n}\left(T^k \mathbf{x}\right) = \mathbf{x} + N\mathbf{a} + \sum_{i=1}^{m} \sum_{k=0}^{N-1} \mathbf{n}\left(T^k \mathbf{x}\right) 1_{\mathcal{D}_i}\left(T^k \mathbf{x}\right). \tag{3}$$

Since $\mathbf{x}$ is a recurrent point for T, there exists an integer sequence $(N_p)_{p\in\mathbb{N}}$ such that $T^{N_p}(\mathbf{x})/N_p$ converges to zero when p tend to infinity. By Birkhoff theorem we obtain in Equation (3) :

$$\frac{T^{N_p}(\mathbf{x})}{N_p} = \frac{\mathbf{x}}{N_p} + \mathbf{a} + \sum_{i=1}^{m} \frac{1}{N_p} \sum_{k=0}^{N_p-1} \mathbf{n}\left(T^k \mathbf{x}\right) 1_{\mathcal{D}_i}\left(T^k \mathbf{x}\right).$$

And when N_p tends to infinity, we get:

$$0 = \mathbf{a} + A_1 \mathbf{r}_1 + \cdots + A_m \mathbf{r}_m, \tag{4}$$

where the vectors $\mathbf{r}_i$ are defined in (1). Since the set $\mathcal{D}$ is of volume 1 we have also:

$$1 = A_1 + \cdots + A_m. \tag{5}$$

Thus Equation (4) can be written $0 = \mathbf{a}' + A_2 \mathbf{r}'_1 + \cdots + A_m \mathbf{r}'_m$, where $\mathbf{r}'_i = \mathbf{r}_i - \mathbf{r}_1$ and $\mathbf{a}' = \mathbf{a} + \mathbf{r}_1$. By Lemma 5.1 we deduce that if $m \leq k$, the translation by vector $\mathbf{a}'$ can not be a minimal translation, thus the translation by vector $\mathbf{a}$ can not be a minimal translation, this is a contradiction.

5.3 Proof of Proposition 3.5

Let $(T, \mathcal{D}_1, \dots, \mathcal{D}_{p_k(1)})$ be a piecewise translation associated to the minimal translation by $\mathbf{a}$ on the torus $\mathbb{T}^k$ and let n be an integer. To each step $n \geq 1$, the $(n-1)$-th refinement of the initial partition consists of $p_k(n)$ domains denoted by $\mathcal{D}_1^{(n)}, \dots, \mathcal{D}_{p_k(n)}^{(n)}$. The measures of domains are respectively denoted by $A_1^{(n)}, \dots, A_{p_k(n)}^{(n)}$. We write $\mathcal{P}^{(n)}$ the partition generated by the domains $\mathcal{D}_1^{(n)}, \dots, \mathcal{D}_{p_k(n)}^{(n)}$.

We fix an integer $n \geq 2$ until the end of the proof.

We define a graph $\mathcal{G}^{(n)}$ as follows :

- To each domain $\mathcal{D}_i^{(n)}$, $i \in \{1, \dots, p_k(n)\}$, is associated a vertex.
- For each integer $\ell \in \{1, \dots, p_k(n+1)\}$, we define an oriented edge $\mathsf{a}_\ell^{(n)}$ from the vertex $\mathcal{D}_i^{(n)}$ to the vertex $\mathcal{D}_j^{(n)}$ if there exists two integers i and j in $\{1, \dots, p_k(n)\}$ such that

$$\mathcal{D}_\ell^{(n+1)} \subset \mathcal{D}_i^{(n)} \text{ and } T^{-1}\big(\mathcal{D}_\ell^{(n+1)}\big) \subset \mathcal{D}_j^{(n)}.$$

This graph is called the n-th *Rauzy Graph* associated to the language of the piecewise translation, see Section 5.5. By assumption, each domain has a non zero Lebesgue measure, and Lebesgue measure is ergodic for the translation, thus the graph $\mathcal{G}^{(n)}$ is connected. Consider the function $f \in C_1(\mathcal{G}^{(n)})$ which associates to any edge $\mathsf{a}_l^{(n)}$ the value $A_l^{(n+1)}$. The applications T and T^{-1} are Lebesgue measure preserving transformations, so the function f is in the space $N(\mathcal{G}^{(n)})$, thus in the space $Z(\mathcal{G}^{(n)})$ by Theorem 4.1. We deduce that there exists $\chi = p_k(n+1) - p_k(n) + 1$ fundamental cycles in this graph such that the function $f \in Z(G_n)$ is a linear combination of fundamental cycles. Thus there exists some real numbers $\alpha_1, \dots, \alpha_\chi$ such that for every integer ℓ there exists a subset $\mathbb{I}_\ell$ of $\{1, \dots, \chi\}$ such that

$$A_\ell^{(n+1)} = \sum_{i \in \mathbb{I}_\ell} \alpha_i. \tag{6}$$

The partition $\mathcal{P}^{(n+1)}$ is a refinement of the initial partition $\mathcal{P}^{(1)}$. So for any integer $\ell \in \{1, \dots, p(1)\}$, there exists a subset $\mathbb{J}_\ell$ of $\{1, \dots, p_k(n+1)\}$ such that

$$\mathcal{D}_\ell = \bigcup_{m \in \mathbb{J}_\ell} \mathcal{D}_m^{(n+1)} \text{ , and so } A_\ell^{(1)} = \sum_{m \in \mathbb{J}_\ell} A_m^{(n+1)}. \tag{7}$$

Using Relations (6) and (7) in Equation (4), we find :

$$\begin{aligned}
0 &= \mathbf{a} + A_1^{(1)}\mathbf{r}_1 + \cdots + A_{p(n)}^{(1)}\mathbf{r}_{p(1)} \\
&= \mathbf{a} + \left(\sum_{m \in \mathbb{J}_1} A_m^{(n+1)}\right)\mathbf{r}_1 + \cdots + \left(\sum_{m \in \mathbb{J}_{p(n+1)}} A_m^{(n+1)}\right)\mathbf{r}_{p(1)} \\
&= \mathbf{a} + \left(\sum_{m \in \mathbb{J}_1} \sum_{i \in \mathbb{I}_m} \alpha_i\right)\mathbf{r}_1 + \cdots + \left(\sum_{m \in \mathbb{J}_{p(n+1)}} \sum_{i \in \mathbb{I}_m} \alpha_i\right)\mathbf{r}_{p(1)}.
\end{aligned}$$

We can reorganize this sum to get the relation:

$$0 = \mathbf{a} + \alpha_1 \cdot \mathbf{s}_1 + \cdots + \alpha_\chi \cdot \mathbf{s}_m,$$

where each vector $\mathbf{s}_i$ is a linear combination with integer coefficients of some vectors $\mathbf{r}_j$, and so is a vector of $\mathbb{Q}^k$. By Lemma 5.1, we conclude that $\chi \geq k-1$, and so $p_k(n+1) - p_k(n) \geq k$.

5.4 Proof of Theorem 3.3

Proposition 3.4 shows that $p_k(1) \geq k+1$. Now Proposition 3.5 shows that for every integer n we have $p_k(n+1) - p_k(n) \geq k$. By induction we deduce that

$$p_k(n) \geq kn + 1.$$

5.5 Rauzy Graphs

We have given a definition of the Rauzy graphs in Section 5.3 with a "geometrically" point of view but we can also introduce them with a purely combinatorial point of view. We refer to [CHM01, Cas97]. Cassaigne studies the graphs of w_2 in [Cas97]. Both definitions are similar. We can deduce from our result :

Proposition 5.2 (Corollary of Theorem 3.3)**.** *The Rauzy graphs of the language of a piecewise translation associated to the minimal translation by* $\mathbf{a}$ *on the torus* $\mathbb{T}^k$ *have an Euler characteristic at least* k.

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